Session 2 Part 1 Financial Options: Basics

Financial Markets and Management

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Bibliographic References

- In any of the standard Corporate Finance textbooks, the chapter on Financial Options or Option Pricing.

OUTLINE

- What are Options?
  - Different Types
    - CALL & PUT
    - European & American
  - Diagrams of Payoffs at Maturity
  - Put-Call Parity.
- Binomial Model:
  - Valuation of options through “replication”;
  - delta of an option and hedging;
  - Volatility;
  - Time to maturity;
  - “Risk Neutral” valuation.
- Black-Scholes Formula.

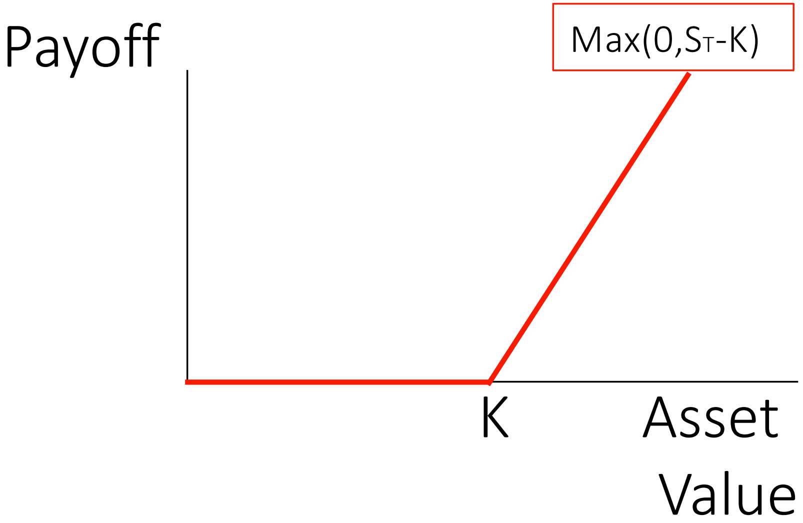
Basic Definitions

- CALL Option:
  - is a right to buy an (underlying) asset at a pre-established exercise (strike) price.
- PUT Option:
  - Is a right to sell an asset at a pre-established exercise price.

- European Options:
  - may be exercised only *at one date* (expiry ou *maturity*).
- American Options:
  - May be exercised any time *until* maturity.

CALL:
Payoffs at Maturity

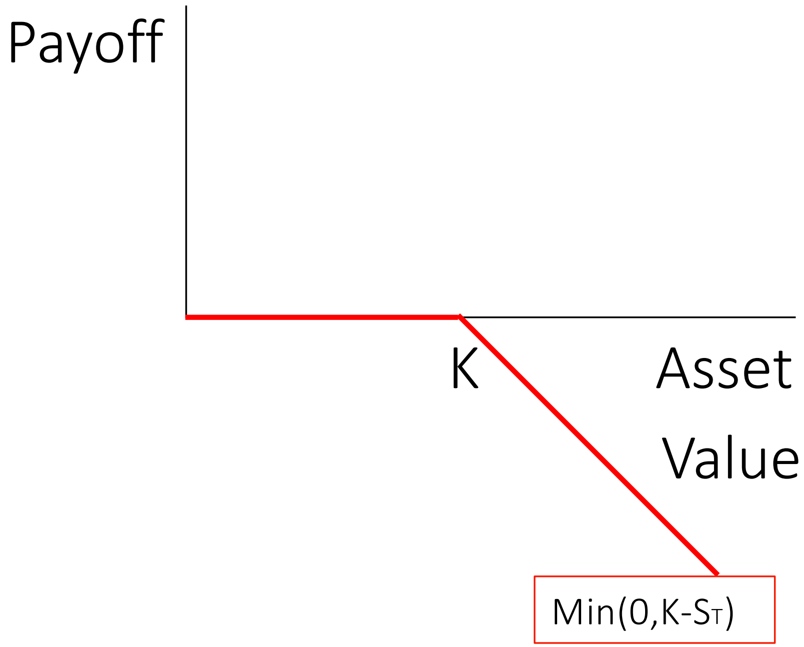
Buyer of the Call (Long):



S_T is asset value at maturity

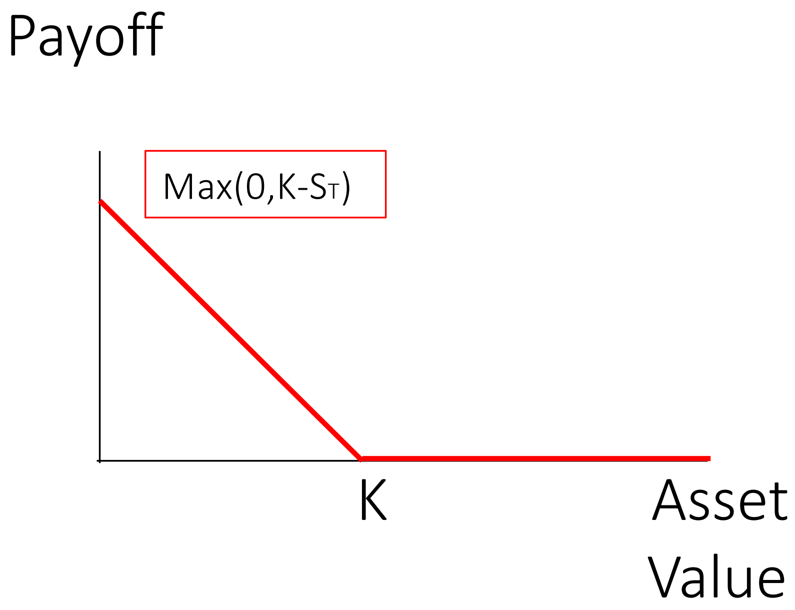
K is the exercise price.

Seller of the Call (Short):



PUT:
Payoffs at Maturity

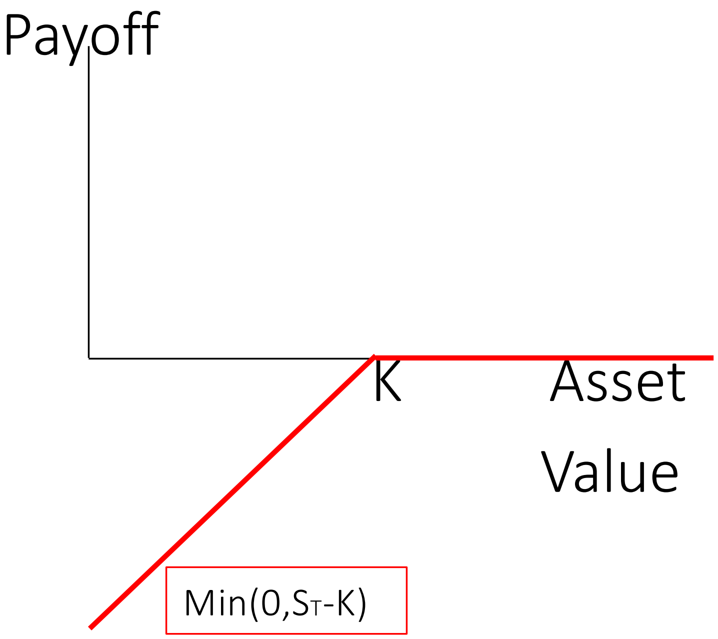
Buyer of the Put (Long):



S_T is asset value at maturity

K is the exercise price.

Seller of the Put (Short):



BINOMIAL VALUATION OF OPTIONS: Example

The Binomial Model assumes that, in each period (time step), the return of the underlying asset can take one of two possible values.

What's the Price of an Option written on such an asset?

(2) Bond (Rf rate 25%)

100

125 (Boom)

125 (Recession)

(1) Share/Underlying Asset

100

145 (Boom "Up")

70 (Recession "Down")

(3) Call Option

(Exercise Price=100)

Option Price

45 (Boom)

0 (Recession)

BINOMIAL MODEL: REPLICATION

- Valuation of Options: using "replication" (hedging portfolio);

- Intuition: find a combination of the stock (underlying asset) and the risk-free bond, which exactly reproduces the payoffs of the option at the maturity.

DELTA OF THE CALL OPTION: WHAT? HOW?

- What? The component of shares (underlying asset) in the replicating portfolio is the delta of the option (Δ);
- Note that a call option is a leveraged position on the stock;
- To compute the delta of the option (Δ), we must solve the following equations:

$$\begin{cases} uS\Delta + (1+r_f)B = C^{up} \\ dS\Delta + (1+r_f)B = C_{down} \end{cases}$$

$$\begin{cases} 1.45 \bullet 100 \bullet \Delta + 1.25 \bullet B = 45 \\ 0.7 \bullet 100 \bullet \Delta + 1.25 \bullet B = 0 \end{cases}$$

- Solving:

$$\Delta = \frac{C^{up} - C_{down}}{(u-d)S}$$

$$B = \frac{uC_{down} - dC^{up}}{(u-d)(1+r_f)}$$

- The value of the option is: $\Delta S + B$

- For this example:

  Δ = 0.6; B = - 33.6 ; The option value is: 0.6*100-33.6=26.4

BINOMIAL MODEL:
REPLICATION - Example, let's check the no arbitrage argument

	Share	Bond	Total
Portfolio	0.6	-33.6	-
Payoff in Boom	0.6*145 = 87	-33.6*1.25 = -42	45
Payoff in Recession	0.6*70 = 42	-33.6*125 = -42	0
Price	100	1	-
Value of the Portfolio	60.0	-33.6	26.4

BINOMIAL MODEL: RISK-NEUTRAL METHOD

- Note: the Risk Neutral valuation derives from the replicating portfolio method;
- Note: the Risk Neutral method is valid for multi-period problems.
- How does it work? By computing the "risk neutral" probabilities of the nodes, and discounting the expected payoffs at the risk-free rate.

- Value of the Call = ΔS+B

May also be written as: $$C = \frac{pC^{up} + (1-p)C_{down}}{r}$$

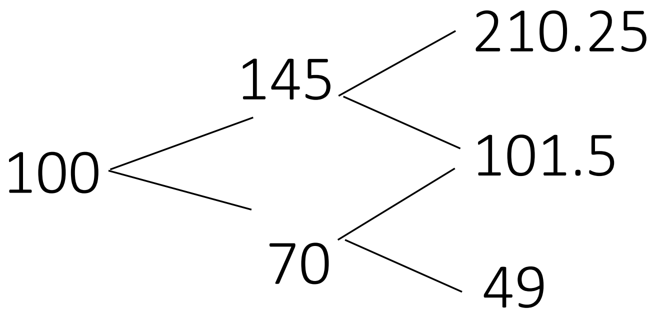
where: $$p = \frac{r-d}{u-d} = \frac{1.25-0.7}{1.45-0.7} = 0.733$$
$$1-p = \frac{u-r}{u-d} = \frac{1.45-1.25}{1.45-0.7} = 0.267$$

- p is the " risk neutral" probability of the boom scenario (and (1-p) of the recession scenario);
- *r* represents the discount factor at the risk-free rate. For example: $(1+r_f)$ or e^{r_f}

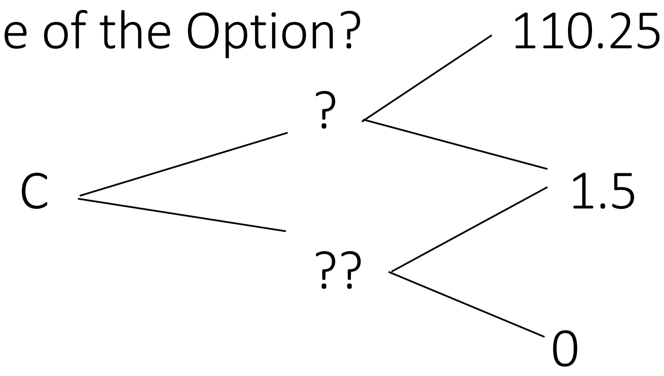
The value of the Call Option is, again, 26.4.

Time to Maturity↑ ⇒ Value of the Call↑

- Share: u=1.45; d=0.7



Value of the Option?



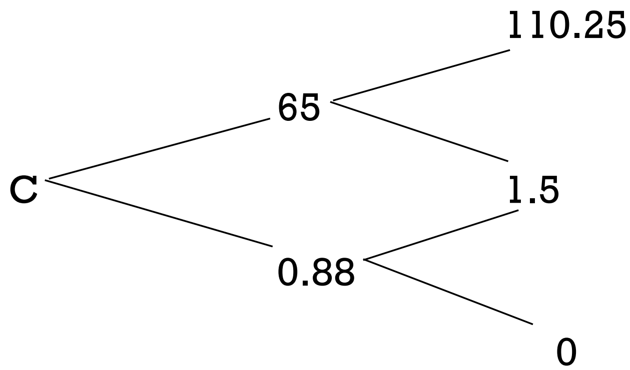
Solve for “?”:

$$? = \frac{pC^{uu} + (1-p)C_{ud}}{r} = \frac{\frac{r-d}{u-d}110.25 + \frac{u-r}{u-d}1.5}{1.25} =$$

$$= \frac{\frac{1.25-0.7}{1.45-0.7}110.25 + \frac{1.45-1.25}{1.45-0.7}1.5}{1.25} = 65$$

?? equals 0.88.

… continues …



- **Finally, the value of the call "today" should be:**

$$C = \frac{pC^{up} + (1-p)C_{down}}{r} = \frac{\frac{r-d}{u-d}65 + \frac{u-r}{u-d}0.88}{1.25} =$$

$$= \frac{\frac{1.25-0.7}{1.45-0.7}65 + \frac{1.45-1.25}{1.45-0.7}0.88}{1.25} = 38.3$$

MOVING TO THE BLACK-SCHOLES MODEL

- In reality, shares may assume many values. Even so, it is possible to “replicate” an option with a portfolio of riskless debt and shares of the underlying asset, for a short interval of time.
- The Black-Scholes model assumes that the rate of return of the underlying asset follows a Random Walk.

BLACK-SCHOLES FORMULA

- Value of the Option = Delta*Price of Share – RiskFree Loan

C = N(d1)* S - N(d2)*PV(K)

with:

- N(d) = Cumulative Normal distribution;
- K = exercise price of call;
- t = time to maturity (in years);
- S = current share price;
- σ = volatility (standard deviation of the rate of return of the underlying asset).
  - NOTE: to establish a "link" between gthe Black-Scholes Annual Volatility and the Binomial parameters u and d, for a "branch movement" of "dt" (for example dt=1 if the branch is 1 year long; dt=1/4 if the branch is 3-months long, you can use: $u = e^{\sigma.\sqrt{dt}}$ and $\mathrm{d} = \frac{1}{u}$.

$$d_1 = \frac{\ln\left(S / PV(K) \right)}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

VOLATILITY ESTIMATION: Example

- Based on historical daily prices, compute the log-returns, and their standard deviation.
- Example:
  - **Annualized Volatility**: multiply Daily Volatility by the square root of the Number of transaction days.
    - It's approximately 250 (or 260?)
- σ = annualized volatility

  = √250*daily volat. = 19.4%

Day	Closing Stock Price Si	Si / Si-1	Daily Return ln(Si/Si-1)
0	20		
1	20,125	1,00625	0,006231
2	19,875	0,987578	-0,0125
3	20	1,006289	0,00627
4	20,5	1,025	0,024693
5	20,25	0,987805	-0,01227
6	20,875	1,030864	0,030397
7	20,875	1	0
8	20,875	1	0
9	20,75	0,994012	-0,00601
10	20,75	1	0
11	21	1,012048	0,011976
12	21,125	1,005952	0,005935
13	20,875	0,988166	-0,0119
14	20,875	1	0
15	21,25	1,017964	0,017805
16	21,375	1,005882	0,005865
17	21,375	1	0
18	21,25	0,994152	-0,00587
19	21,75	1,023529	0,023257
20	22	1,011494	0,011429
STD. DEV			0,012308

PUT OPTIONS: BINOMIAL MODEL

(1) Share

100 → 145(Boom)
100 → 70(Recession)

(2) Bond (Rf 25%)

100 → 125 (Boom)
100 → 125 (Recession)

(3) Put Option (K = 100)

Option Price → 0 (Boom)
Option Price → 30 (Recession)

What's the price of the Put Option?

PUT-CALL PARITY

- Put-Call Parity for European options, with the same exercise price, same time to maturity, and same underlying asset (and no dividend payment)

$$C - P = S - PV(K)$$

In the 1-period example:

Share Price = 100

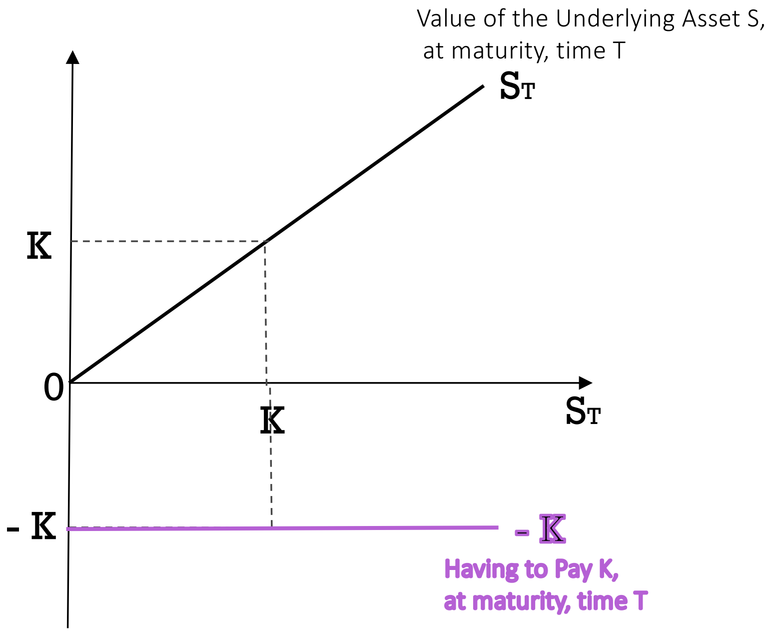
PV(K) = 100/1.25 = 80

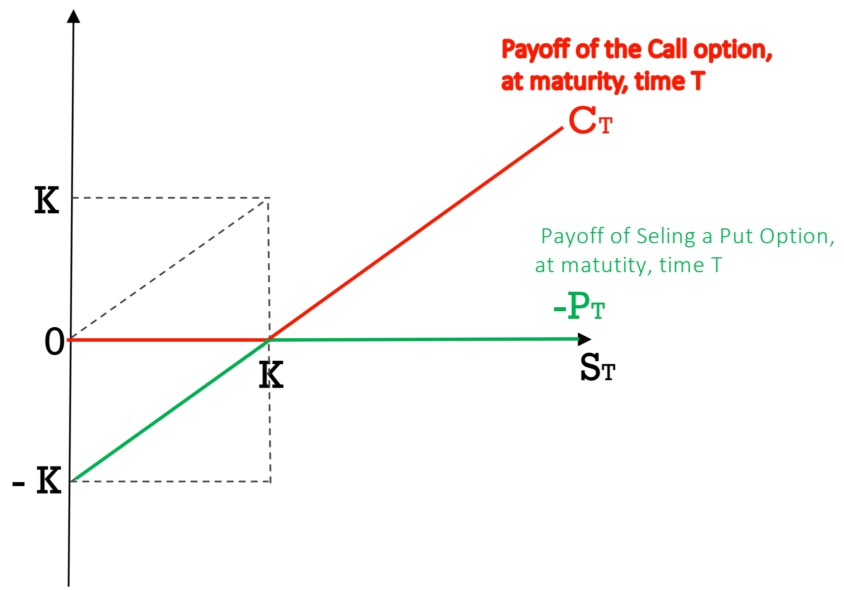
Call Price = 26.4

Put Price = 6.4

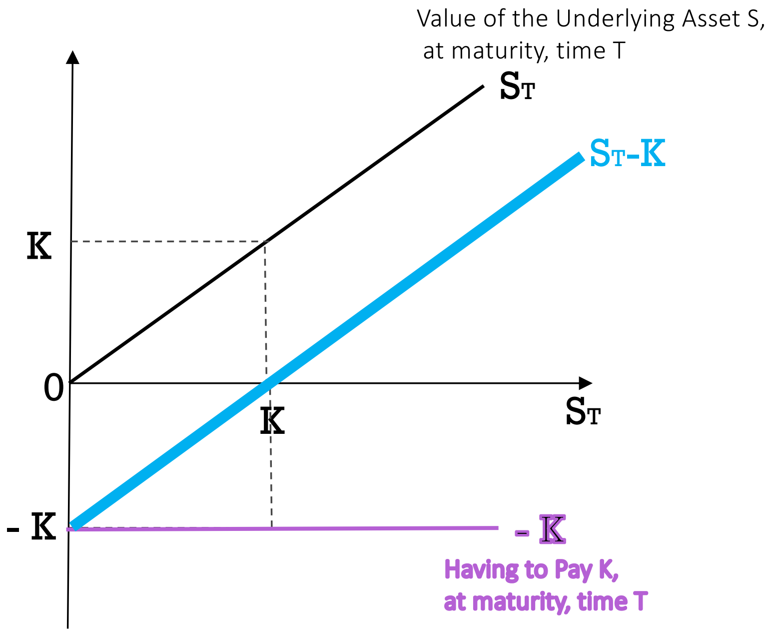
NOTE: To compute the value of the Put Option, can also build the replicating/hedging portfolio or apply the Risk-Neutral Vauation Method.

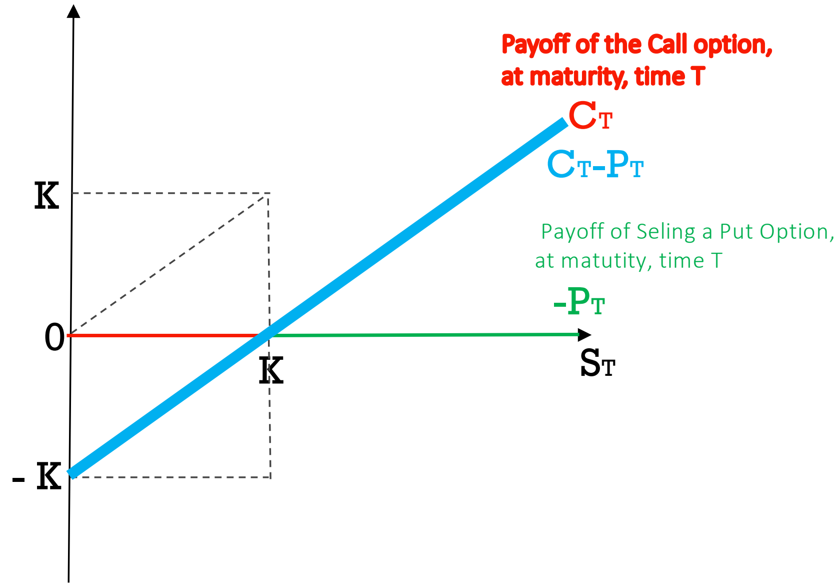
PUT-CALL PARITY





PUT-CALL PARITY





At maturity T: $C_T - P_T = S_T - K$

So, at any time t before maturity: $C_t - P_t = S_t - PV_t(K)$

Generally Speaking: $C - P = S - PV(K)$

MAIN DETERMINANTS OF THE VALUE OF AN OPTION

	Call	Put
Current Stock price	↑	↓
Exercise Price	↓	↑
Time to maturity	**	**
Stock Volatility	↑	↑
Interest Rate	↑	↓
Cash Dividends	↓	↑

↑ Means that the value of the option increases when this variable goes up;

↓ Means that the value of the option decreases when the value of this variable goes up;

** means that the effect is ambiguous for european options.

AMERICAN VERSUS EUROPEAN: CALLS

Think of an American Call Option.

Early Exercise implies:

- Gain:
  - Dividend;
- Loss:
  - Interest in the Exercise Price paid;
  - Option.

- Therefore:
  - Non-Dividend-Paying Shares:
    - It is never optimal to exercise early;
    - American and European are worth the same.
  - Dividend-Paying Shares:
    - American Call $\geq$ European Call

AMERICAN VERSUS EUROPEAN: PUTS

- Think of an American Put Option.
- Early Exercise implies:
  - Gain:
    - Interest on the exercise price.
  - Loss:
    - Dividend;
    - Option.

Hence:

- Exercising Early may be optimal even if there are no dividends.
- American Put $\geq$ European Put

APPENDIX Tools & Tricks

1. TRICKS TO MOVE FROM THE CONTINUOUS TIME WORLD OF BLACK SCHOLES TO THE BINOMIAL MODEL
2. HOW TO USE THE BINOMIAL MODEL WITH DIFFERENT TIME STEPS dt
3. STEPS TO FOLLOW IN TO APPLY THE BINOMIAL MODEL (considering continuous docounting rate Rf)

Appendix 1,2

1. FROM BS TO BINOMIAL

- Given the Black Scholes Volatilty Parameter Sigma σ
- And Continuous Discounting at Risk Free Rate Rf

2. HOW TO USE THE BINOMIAL MODEL WITH DIFFERENT TIME STEPS dt

- For the Binomial Valuation you must know first the “Time Step” dt for each “jump in the Tree”:
  - Given dt
  - Compute $u = e^{\sigma\sqrt{dt}}$
  - Compute $d = \frac{1}{u}$
  - Compute $p = \frac{e^{Rf.dt} - d}{u-d}$
  - Note: If you want another valuation with a different time step dt, then you must recompute u, d and p.

Appendix
3

3. STEPS TO FOLLOW TO USE THE BINOMIAL MODEL

1. **Given dt, Sigma and Rf**, calculate:
   - Compute $u = e^{\sigma\sqrt{dt}}$
   - Compute $d = \frac{1}{u}$
   - Compute $p = \frac{e^{Rf.dt} - d}{u - d}$

2. Build the Tree for the Underlying Asset S

3. Compute the Tree of the Option, starting with the Payoffs at Maturity.

4. Move backwards every period, by discounting the payoffs **of the Option in final period T to period (T-1) with risk neutral probability p and risk-free rate Rf,** and repeat the procedure.

5. Until you get the Value of the Option at time 0.

Example

- Suppose you know $R_F = 2\%\ with\ continuous\ compounding, S_0 = 100, \sigma = 30\%.$
- There is a Call Option with $K = 95, time\ to\ maturity\ T = 1\ year$
- You want to use the Binomial Model with time step $dt = \frac{1}{2}$, with jumps of 1 semester.

1. Given dt = 0.5, compute u, d, p:

$$u = e^{\sigma\sqrt{dt}} = e^{0.3\sqrt{0.5}} = 1.2363111$$

$$d = \frac{1}{u} = \frac{1}{1.2363111} = 0.8088579$$

$$p = \frac{e^{Rf.dt} - d}{u - d} = \frac{e^{0.02*0.5} - 0.8088579}{1.2363111 - 0.8088579} = 0.4706767$$

2. Build the Tree for the Underlying Asset S:

t=0 ⟵ dt=0.5 ⟶ t=0.5 ⟵ dt=0.5 ⟶ t=1

$S_0 = 100$

$S^u = uS_0 = 1.236311 \times 100$
$= 123.6311$

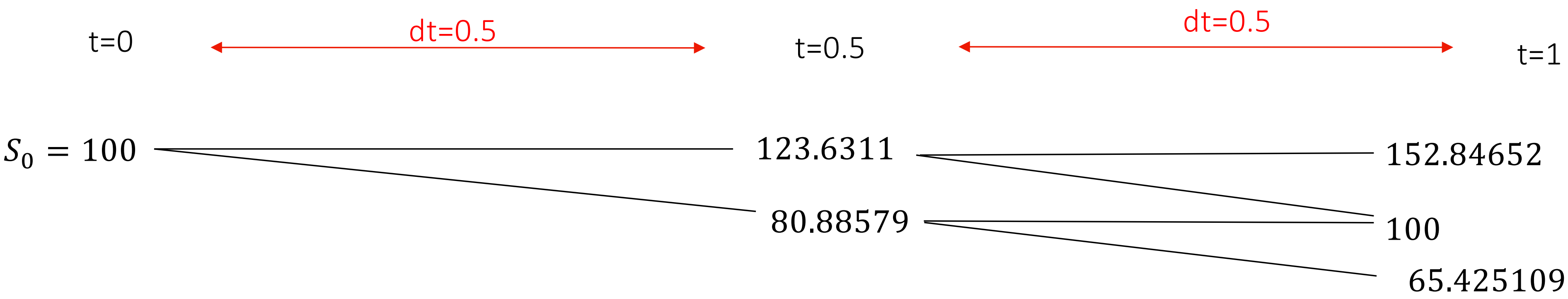
$S^d = dS_0 = 0.8088579 \times 100$
$= 80.88579$

$S^{uu} = 1.2361 \times 123.6311$
$= 152.84652$

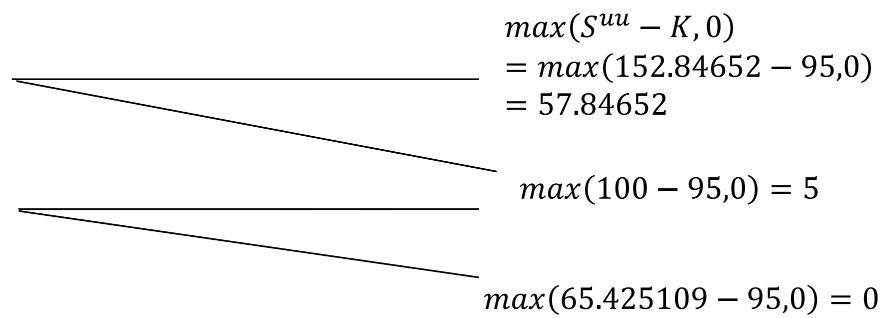
$S^{ud} = S^{du}$
$= 0.80885 \times 123.6311$
$= 1.236311 \times 80.88579$
$= 100$

S^{dd}
$= 0.8088579 \times 80.88579$
$= 65.425109$

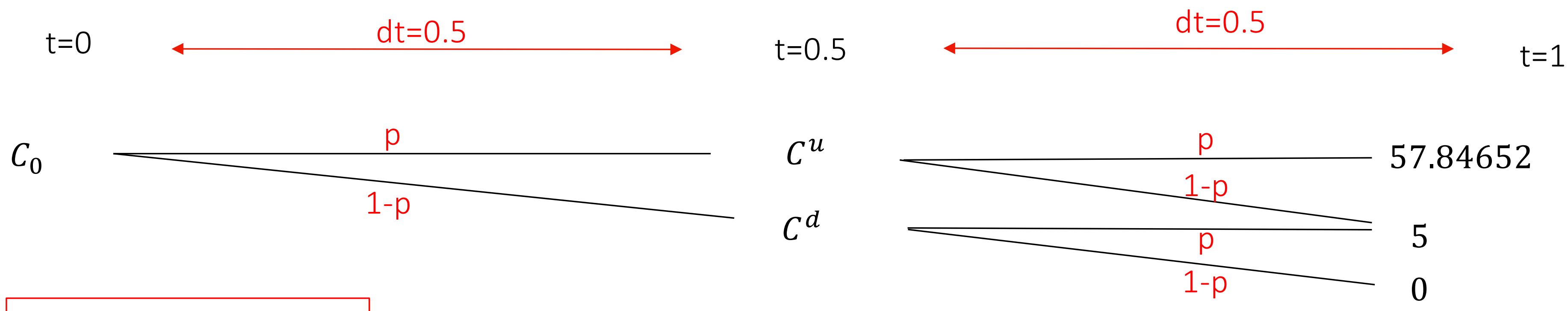
3. Compute the Payoffs at Maturity for the Option:



Payoffs at Maturity for the Option:



4. Discount Back Payoffs at Maturity of the Option:



Remember:

$$R_F = 2\%$$
$$p = 0.4706767$$
$$1 - p = 0.5293233$$

$$C^u = e^{-Rf\times dt}\left(p\times C^{uu} + (1-p)C^{ud}\right)$$
$$= e^{-0.02\times 0.5}(0.4706767\times 57.84652 + 0.5293233\times 5) = 29.5763768$$

$$C^d = e^{-0.02\times 0.5}(0.4706767\times 5 + 0.5293233\times 0) = 2.32996699$$

$$C_0 = e^{-0.02\times 0.5}(0.4706767\times 29.5763768 + 0.5293233\times 2.32996699) = 15.0034305$$

Rf	2%	
Sigma	30%	
So	100	
K	95	
T	1	
dt	0,5	
u	1,23631111	
d	0,80885789	
p	0,47067671	
1-p	0,52932329	
TREE UNDERLYING ASSET S		
t=0	0,5	1
100	123,631111	152,846516
	80,8857893	100
		65,4251092
TREE CALL OPTION		
15,0034305	29,5763768	57,846516
	2,32996699	5
		0

Microsoft Excel Worksheet